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Preconditioners for inhomogeneous anisotropic problems in spherical geometry

D. E. Brown[1,‡], N. K. Nichols[1,*,†] and M. J. Bell[2]

[1] *Department of Mathematics, The University of Reading, Whiteknights, Box 220, Berkshire RG6 6AX, U.K.*
[2] *Met Office, FitzRoy Road, Exeter, Devon EX1 3PB, U.K.*

SUMMARY

Elliptic equations arising in free-surface ocean models are typically solved using iterative methods. Anisotropy associated with standard spherical co-ordinate systems causes the convergence of the iterative methods to be slow, particularly in polar regions. This behaviour is demonstrated here using a diagonally preconditioned conjugate gradient method (PCG) method to solve a Helmholtz elliptic model problem with a standard five-point discretization scheme on a 2D spherical domain. The cause of the poor polar convergence is shown to be the increased importance, with increased mesh anisotropy, of eigenmodes with strong polar signals. Block diagonal and alternating direction implicit (ADI) preconditioners are found to give improved convergence. Crown copyright 2005. Reproduced with the permission of Her Majesty's Stationery Office. Published by John Wiley & Sons, Ltd.

KEY WORDS: anisotropic; elliptic; preconditioning; alternating direction implicit; finite-difference

1. INTRODUCTION

Most ocean models in use today are based on integrating the incompressible primitive equations on a sphere. Complex topography is used at the ocean bottom and the ocean surface is either fixed or free to move with time. The ocean basins themselves typically contain irregularly shaped coastlines and islands which require the inclusion of specific boundary conditions into any solution algorithm.

The equations that arise in the free-surface models are elliptic and isotropic. They are rendered mesh anisotropic by the use of spherical co-ordinates. An operator is anisotropic if

*Correspondence to: N. K. Nichols, Departments of Mathematics and Meteorology, The University of Reading, P.O. Box 220, Whiteknights, RG6 6AX Reading, U.K.
†E-mail: n.k.nichols@rdg.ac.uk
‡E-mail: smr00deb@reading.ac.uk

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its local properties vary with direction. As an example, we consider a Helmholtz equation

$$-\frac{\partial}{\partial\lambda}\left(L^{\lambda}\frac{\partial U}{\partial\lambda}\right)-\frac{\partial}{\partial\phi}\left(L^{\phi}\frac{\partial U}{\partial\phi}\right)+kU=\gamma(\lambda,\phi) \tag{1}$$

discretized on a sphere. If the coefficients are such that L^{λ} is much larger than L^{ϕ}, then the discrete operator is poorly conditioned. This is an example of strong anisotropy. In the ocean models, the effects of mesh anisotropy are seen in the latitudinally varying rates of convergence of the iterative methods. Poor rates of convergence are observed in polar regions. In these cases, $L^{\lambda}\approx L^{\phi}$ in equatorial regions, but $L^{\lambda}\gg L^{\phi}$ in polar regions. This is an example of inhomogeneous anisotropy. In these cases, an operator is inhomogeneous if its properties, when measured in a particular direction, change with location. Significantly higher errors in the iterates are observed in the polar regions than in equatorial and mid-latitude regions.

The main aim of this paper is to explain how the mesh anisotropy of the elliptic problem causes the poor polar convergence in the PCG method. The standard diagonal preconditioner is used for illustration. As alternatives, block diagonal and alternating direction implicit (ADI) preconditioners are also considered and their impact on the convergence speeds and conditioning are investigated.

2. FREE-SURFACE OCEAN MODEL

Many of the ocean general circulation models currently in use are based on the Bryan–Cox–Semtner (henceforth BCS) model initially introduced by Bryan [3] in the late 1960s and later modified by Cox [4] and Semtner [10]. The BCS model solves the primitive equations in a spherical co-ordinate system using hydrostatic and Boussineq approximations. The formulation of the BCS model we investigate here is the implicit free-surface barotropic formulation introduced by Dukowicz [5] and summarized briefly as follows. The barotropic, or vertically averaged, equations of state are given by

$$\begin{aligned}\frac{\partial u}{\partial t}-fv&=-g\frac{1}{a\cos\phi}\frac{\partial\eta}{\partial\lambda}+G^{x}\\ \frac{\partial v}{\partial t}+fu&=-g\frac{1}{a}\frac{\partial\eta}{\partial\phi}+G^{y}\\ \frac{\partial\eta}{\partial t}+\frac{1}{a\cos\phi}&\left[\frac{\partial Hu}{\partial\lambda}+\frac{\partial Hv\cos\phi}{\partial\phi}\right]=0\end{aligned} \tag{2}$$

where λ and ϕ are longitude and latitude, respectively, η is the free-surface height, f is the Coriolis parameter, g is the gravitational acceleration constant, $H=H(\lambda,\phi)$ is the total depth of the ocean, $\mathbf{u}=(u,v)$ are the barotropic velocity components, and G^{x}, G^{y} represent baroclinic forcing. Dukowicz [5] proposes the general time discretization of Equation (2)

$$\begin{aligned}\frac{u^{n+1}-u^{n-1}}{2\tau} - f v^{\alpha'} &= -g\frac{1}{a\cos\phi}\frac{\partial \eta^{\alpha}}{\partial \lambda} + G^{x,n}\\ \frac{v^{n+1}-v^{n-1}}{2\tau} + f u^{\alpha'} &= -g\frac{1}{a}\frac{\partial \eta^{\alpha}}{\partial \phi} + G^{y,n}\\ \frac{\eta^{n+1}-\eta^{n}}{\tau} + \frac{1}{a\cos\phi}&\left[\frac{\partial H u^{\theta}}{\partial \lambda} + \frac{\partial H v^{\theta}\cos\phi}{\partial \phi}\right] = 0\end{aligned} \tag{3}$$

with

$$\begin{aligned}\mathbf{u}^{\alpha'} &= \alpha'\mathbf{u}^{n+1} + (1-\alpha'-\gamma')\mathbf{u}^{n} + \gamma'\mathbf{u}^{n-1}\\ \eta^{\alpha} &= \alpha\eta^{n+1} + (1-\alpha-\gamma)\eta^{n} + \gamma\eta^{n-1}\\ \mathbf{u}^{\theta} &= \theta\mathbf{u}^{n+1} + (1-\theta)\mathbf{u}^{n}\end{aligned} \tag{4}$$

where τ is the fixed timestep, n is the current time level and α, α', γ, γ' and θ are coefficients used to parameterize the time centreing of the pressure gradient, Coriolis, and divergence terms. Eliminating u^{n+1} and v^{n+1} in (3) and rearranging, we obtain an implicit equation for η' that represents the change in free-surface height, η, between two consecutive timesteps of the free-surface ocean model. The elliptic operator, which is solved at every timestep, is of the form

$$-\frac{1}{a\cos\phi}\left[\frac{\partial}{\partial\lambda}\left(\frac{H(\lambda,\phi)}{a\cos\phi}\frac{\partial\eta'}{\partial\lambda}\right) + \frac{\partial}{\partial\phi}\left(\frac{H(\lambda,\phi)\cos\phi}{a}\frac{\partial\eta'}{\partial\phi}\right)\right] + \beta\eta' = S(\lambda,\phi) \tag{5}$$

where $\beta = 1/(2\alpha\theta g\tau)$, and $S(\lambda,\phi)$ is a nonlinear forcing term.

3. PRECONDITIONERS FOR MODEL PROBLEM

We consider a limited area, northern hemisphere, Helmholtz model problem of the following form in our numerical experiments

$$\begin{gathered}-\frac{1}{\cos\phi}\left[\frac{\partial}{\partial\lambda}\left(\frac{1}{\cos\phi}\frac{\partial U}{\partial\lambda}\right) + \frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial U}{\partial\phi}\right)\right] + kU = \gamma(\lambda,\phi)\\ U(0°E,\phi)=0, \quad U(30°E,\phi)=0\\ U(\lambda,10°N)=0, \quad U(\lambda,l)=0\\ \lambda\in(0°E,30°E), \quad \phi\in(10°N,l), \quad l\in(40°N,89.5°N)\end{gathered} \tag{6}$$

where $k\geqslant 0$ (we use $k=0.01$ in the displayed results), and γ is known. This is a simplified version of (5) containing the essential features of the problem. In our experiments we investigate the effects of moving the northern boundary of the domain closer to the pole. We discretize the problem using a standard five-point finite difference scheme with a constant step size h in both directions and taking a natural ordering of the grid points. This gives rise to a matrix equation of the general form

$$A\mathbf{U} = \mathbf{b} \tag{7}$$

where the variable $\mathbf{U}$ is a (unknown) column vector of the grid point values of the variable U and $\mathbf{b}$ is a (known) column vector representing boundary values and source terms. The system matrix A is a real, symmetric, $m \times m$ matrix representing the discretized model equations (where m is the number of grid points). It is also square, sparse, irreducible and diagonally dominant with strict diagonal dominance in at least one row. It is therefore irreducibly diagonally dominant and hence positive-definite [11].

Free-surface ocean models typically use a PCG method with a preconditioner containing only the diagonal elements of A to solve (7). The PCG method may be thought of as an acceleration procedure for a stationary method with iteration matrix $G_P = I - P^{-1}A$, where P is the preconditioner [7]. The diagonally preconditioned iteration matrix is given by $G_D = I - D^{-1}A$, where $D = \mathrm{diag}(A)$, whilst the block preconditioned method uses $P = B = \mathrm{blockdiag}(A)$, with each diagonal block consisting of all points on one latitude in the grid. The ADI preconditioned iteration matrix, G_{ADI} [11] is similar to

$$(\Upsilon I - H_\Upsilon)(\Upsilon I + H_\Upsilon)^{-1}(\Upsilon I - V_\Upsilon)(\Upsilon I + V_\Upsilon)^{-1} \tag{8}$$

where

$$\begin{aligned}(H_\Upsilon U)(\lambda_i, \phi_j) &= -U(\lambda_i + h, \phi_j) + 2U(\lambda_i, \phi_j) - U(\lambda_i - h, \phi_j) + kU(\lambda_i, \phi_j)/2 \\ (V_\Upsilon U)(\lambda_i, \phi_j) &= -U(\lambda_i, \phi_j + h) + 2U(\lambda_i, \phi_j) - U(\lambda_i, \phi_j - h) + kU(\lambda_i, \phi_j)/2\end{aligned} \tag{9}$$

and Υ is a free parameter. The matrices defined in (9) have the following properties: H and V are Stieltjes matrices and are diagonally dominant and positive-definite with positive

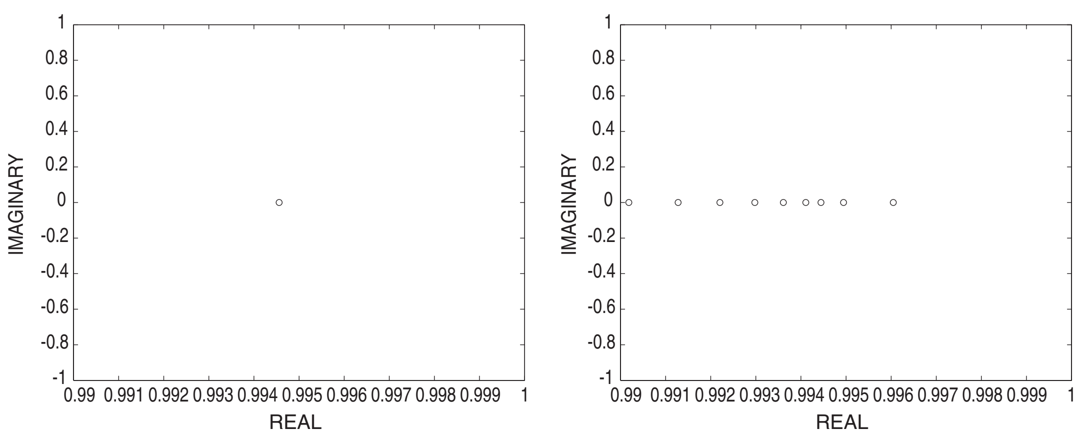

Figure 1. Leading eigenvalues of G_D with real parts greater than 0.99 plotted on the complex plane for cases $l = 40°$ and $89°$, respectively, $h = 1°$.

diagonal entries and non-positive non-diagonal entries. The preconditioning step involves one sweep of the ADI scheme. The convergence of the PCG method with preconditioned iteration matrices $G(D)$, $G(B)$, $G(\text{ADI})$ for solving the discrete approximations (7) to problem (6) is established in Reference [2] using results from References [1, 6–9, 11].

4. NUMERICAL EXPERIMENTS

Figure 1 shows the effect that the increased anisotropy due to moving the northern boundary closer to the pole has on the eigenvalues of G_D. We observe the clustering of secondary eigenvalues of G_D near the slightly larger, leading eigenvalue. The errors projected onto the

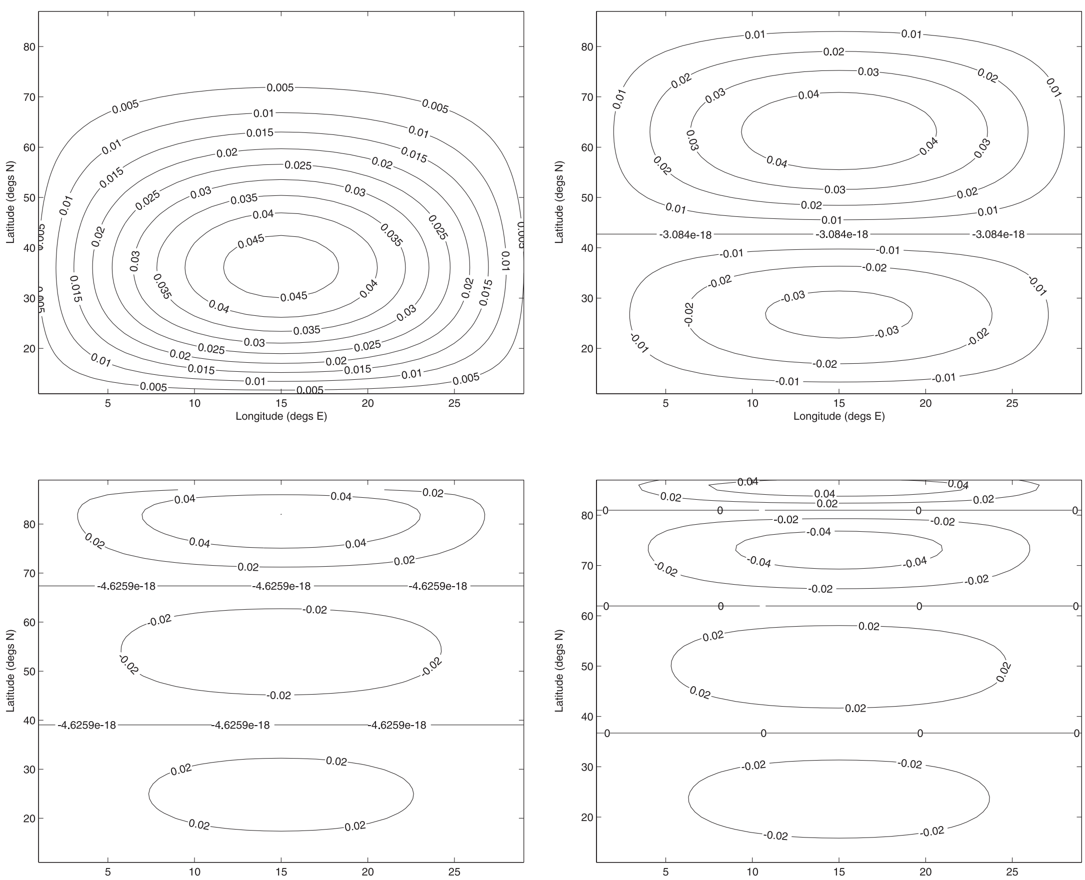

Figure 2. Leading four eigenvectors of G_D for the case $l = 88°$, $h = 1°$, plotted for a $30°$ segment of longitude (horizontal axis) vs. latitude (vertical axis) on the unit sphere.

Table I. Variation of condition number with varying northern boundary.

	$\kappa(A)$		
Boundary	$h = \frac{1}{2}^\circ$	$h = 1^\circ$	$h = 2^\circ$
40°	2.19×10^3	544.18	134.17
70°	4.28×10^3	1.04×10^3	249.57
88°	3.12×10^4	6.51×10^3	1.21×10^3
89°	5.20×10^4	9.75×10^3	NA

Table II. Variation of condition number and spectral radii with preconditioners, $l = 88^\circ$, $h = 1^\circ$.

Preconditioner	$\kappa(P^{-1}A)$	$\rho(G_P)$
None	6.51×10^3	—
Diagonal	691.92	0.9960
Block	293.92	0.9901
ADI	139.31	0.9601

eigenvectors associated with these eigenvalues will converge more slowly as the anisotropy increases. Figure 2 shows the leading four eigenvectors of G_D. We see that the lead eigenmode does not possess a significant signal in the polar region and for this reason, as the boundary is moved northwards, the spectral radius of G_D varies little. It is the secondary eigenmodes that become more significant with increased anisotropy and therefore converge more slowly. This causes the poor polar convergence.

Tables I and II show the effects on the conditioning of the problem of the increased anisotropy due to moving the northern boundary closer to the pole, and of the use of the different preconditioners. We observe that moving the boundary nearer to the pole causes the conditioning to increase by more than an order of magnitude. We also observe that the conditioning of the matrix with the block preconditioner is better than with the diagonal preconditioner, with ADI preconditioning giving a further improvement on that. We would therefore expect ADI preconditioning and, to a lesser extent, block preconditioning to yield better convergence rates than the diagonal preconditioner. This can, indeed, be seen in Figure 3. The figure shows the residual errors after a fixed amount of CPU time, with ADI and block preconditioning clearly superior.

5. CONCLUSIONS

Our analysis of the eigenvectors and eigenvalues of the iteration matrix G_D of our iterative method leads us to conclude that the problem of larger residual errors in polar regions of our model is due to the increased importance, with increased mesh anisotropy, of 'nearly' leading

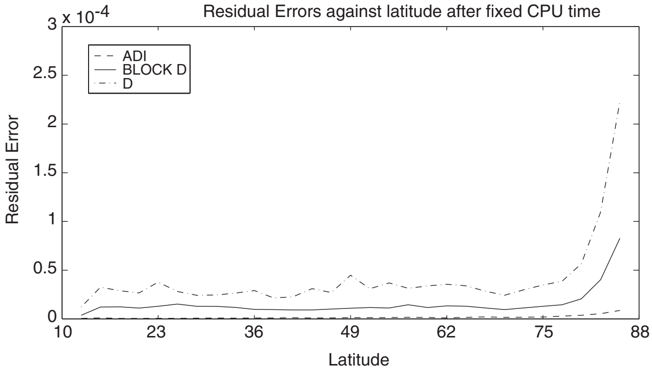

Figure 3. Latitudinal variance in convergence for limited area Helmholtz problem with diagonal, block and ADI preconditioners, $l = 88°$, $h = 1°$.

eigenvectors with significant signals in polar regions. Our numerical experiments have shown that block Jacobi and particularly ADI preconditioners can yield considerable reductions in the large residual errors in polar regions and hence improve the overall convergence speeds of the elliptic solvers on the sphere. Further details of this work may be found in Reference [2].

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